ELSEVIER
Journal of Geometry and Physics 36 (2000) 140–151
JOURNAL OF GEOMETRY AND PHYSICS

Symmetry reduction of discrete Lagrangian mechanics on Lie groups

Jerrold E. Marsden [a], Sergey Pekarsky [a,*], Steve Shkoller [b]

[a] *Control and Dynamical Systems, California Institute of Technology, 107-81 Pasadena, CA 91125, USA*
[b] *Department of Mathematics, University of California, Davis, CA 95616, USA*

Received 14 October 1999; received in revised form 7 February 2000

Abstract

For a discrete mechanical system on a Lie group G determined by a (reduced) Lagrangian ℓ, we define a Poisson structure via the pull-back of the Lie–Poisson structure on the dual of the Lie algebra $\mathfrak{g}^*$ by the corresponding Legendre transform. The main result shown in this paper is that this structure coincides with the reduction under the symmetry group G of the canonical discrete Lagrange 2-form $\omega_{\mathbb{L}}$ on $G \times G$. Its symplectic leaves then become dynamically invariant manifolds for the reduced discrete system. Links between our approach and that of groupoids and algebroids as well as the reduced Hamilton–Jacobi equation are made. The rigid body is discussed as an example.

MSC: 70H35; 70E15; 58F

Subj. Class.: Dynamical systems

Keywords: Euler–Poincaré; Symplectic; Poisson

1. Introduction

Background. This paper continues our development of discrete Lagrangian mechanics on a Lie group introduced in Ref. [7]. In our earlier paper, using the context of the Veselov method for discrete mechanics, discrete analogues of Euler–Poincaré and Lie–Poisson reduction theory (see, e.g., [5]) were developed for systems on finite-dimensional Lie groups G with Lagrangians $L : TG \to \mathbb{R}$ that are G-invariant. The resulting discrete equations provide "reduced" numerical algorithms which manifestly preserve the symplectic structure. The manifold $G \times G$ is used as the discrete approximation of TG, and a discrete

* Corresponding author.
E-mail addresses: marsden@cds.caltech.edu (J.E. Marsden), sergey@cds.caltech.edu (S. Pekarsky), shkoller@math.ucdavis.edu (S. Shkoller).

0393-0440/00/$ – see front matter © 2000 Published by Elsevier Science B.V.
PII: S0393-0440(00)00018-8

Lagrangian $\mathbb{L} : G \times G \to \mathbb{R}$ is constructed from a given Lagrangian L in such a way that the G-invariance property is preserved. Reduction by G results in a new "variational" principle for the reduced Lagrangian $\ell : G \cong (G \times G)/G \to \mathbb{R}$, which then determines the discrete Euler–Poincaré (DEP) equations. Reconstruction of these equations is consistent with the usual Veselov discrete Euler–Lagrange (DEL) equations developed in Refs. [6,13], which are naturally symplectic-momentum algorithms. Furthermore, the solution of the DEP algorithm leads directly to a discrete Lie–Poisson (DLP) algorithm. For example, when $G = \mathfrak{so}(n)$, the DEP and DLP algorithms for a particular choice of the discrete Lagrangian $\mathbb{L}$ are equivalent to the Moser–Veselov [9] scheme for the generalized rigid body.

Main results of this paper. We show that when a discrete Lagrangian $\mathbb{L} : G \times G \to \mathbb{R}$ is G-invariant, a Poisson structure on (a subset) of one copy of the Lie group G can be defined which governs the corresponding discrete reduced dynamics. The symplectic leaves of this structure become dynamically invariant manifolds which are manifestly preserved under the structure preserving DEP algorithm (see Section 2.1).

Moreover, starting with a DEP system on G one can readily recover, by means of the Legendre transformation, the corresponding Lie–Poisson Hamilton–Jacobi system on $\mathfrak{g}^*$ analyzed by Ge and Marsden [1]; the relationship between the DEL and DEP equations and the Lie–Poisson Hamilton–Jacobi equations was examined from a different point of view in our companion paper [7].

We also apply Weinstein's results on Lagrangian mechanics on groupoids and algebroids [12] to the setting of regular Lie groups. The groupoid–algebroid setting reveals new and interesting connections between discrete and continuous dynamics.

2. Discrete reduction

In this section we review the DEP reduction of a Lagrangian system on $G \times G$ considered in detail in Ref. [7]. We approximate TG by $G \times G$ and form a discrete Lagrangian $\mathbb{L} : G \times G \to \mathbb{R}$ from the original Lagrangian $L : TG \to \mathbb{R}$ by

$$\mathbb{L}(g_k, g_{k+1}) = L(\kappa(g_k, g_{k+1}), \chi(g_k, g_{k+1})),$$

where κ and χ are functions of (g_k, g_{k+1}) which approximate the current configuration $g(t) \in G$ and the corresponding velocity $\dot{g}(t) \in T_g G$. We choose discretization schemes for which the discrete Lagrangian $\mathbb{L}$ inherits the symmetries of the original Lagrangian L: $\mathbb{L}$ is G-invariant on $G \times G$ whenever L is G-invariant on TG. In particular, the induced right (left) lifted action of G onto TG corresponds to the diagonal right (left) action of G on $G \times G$.

Having specified the discrete Lagrangian, we form the *action sum*

$$\mathbb{S} = \sum_{k=0}^{N-1} \mathbb{L}(g_k, g_{k+1}),$$

which approximates the action integral $S = \int L \, \mathrm{d}t$, and obtain the DEL equations

$$\mathrm{D}_2\mathbb{L}(g_{k-1}, g_k) + \mathrm{D}_1\mathbb{L}(g_k, g_{k+1}) = 0, \tag{2.1}$$

as well as the discrete symplectic form $\omega_{\mathbb{L}}$, given in coordinates on $G \times G$ by

$$\omega_{\mathbb{L}} = \frac{\partial^2 \mathbb{L}}{\partial g_k^i \partial g_{k+1}^j} \mathrm{d}g_k^i \wedge \mathrm{d}g_{k+1}^j.$$

In Eq. (2.1), D_1 and D_2 denote derivatives with respect to the first and second argument, respectively. The algorithm (2.1) as well as $\omega_{\mathbb{L}}$ are obtained by extremizing the action sum $\mathbb{S} : G^{N+1} \to \mathbb{R}$ with arbitrary variations. Using this variational point of view, it is known that the flow $\mathbb{F}_t$ of the DEL equations preserves this discrete symplectic structure. This result was obtained using a discrete Legendre transform and a direct computation in Refs. [10,11,13] and a proof using the variational structure directly was given in Ref. [6].

Remark 2.1. *We remark that the discrete symplectic structure $\omega_{\mathbb{L}}$ is not globally defined, but rather need only be nondegenerate in a neighborhood of the diagonal Δ in $G \times G$, i.e. whenever g_k and g_{k+1} are nearby. Section* 3 *of Ref.* [6] *shows that $\omega_{\mathbb{L}}$ arises from the boundary terms of the discrete action sum restricted to the space of solutions of the DEL equations; an implicit function theorem argument relying on the regularity of the discrete Lagrangian $\mathbb{L}$ is required in order to obtain solutions to the DEL equations, and this regularity need only hold in a neighborhood of the diagonal $\Delta \subset G \times G$.*

2.1. The DEP algorithm

The discrete reduction of a right-invariant system proceeds as follows (see [7] for details). The case of left invariant systems is similar. Of course, some systems such as the rigid body are left invariant.

The induced group action on $G \times G$ by an element $\bar{g} \in G$ is simply right multiplication in each component

$$\bar{g} : (g_k, g_{k+1}) \mapsto (g_k \bar{g}, g_{k+1} \bar{g})$$

for all $g_k, g_{k+1} \in G$.

The quotient map is given by

$$\pi_{\mathrm{d}} : G \times G \to \frac{G \times G}{G} \cong G, \qquad (g_k, g_{k+1}) \mapsto g_k g_{k+1}^{-1}. \tag{2.2}$$

One may alternatively use $g_{k+1} g_k^{-1}$ instead of $g_k g_{k+1}^{-1}$ as the quotient map; the projection map (2.2) defines the *reduced discrete Lagrangian* $\ell : G \to \mathbb{R}$ for any G-invariant $\mathbb{L}$ by $\ell \circ \pi_{\mathrm{d}} = \mathbb{L}$, so that

$$\ell(g_k g_{k+1}^{-1}) = \mathbb{L}(g_k, g_{k+1}),$$

and the *reduced action sum* is given by

$$s = \sum_{k=0}^{N-1} \ell(f_{kk+1}),$$

where $f_{kk+1} \equiv g_k g_{k+1}^{-1}$ denotes points in the quotient space. A reduction of the DEL equations results in the *discrete Euler–Poincaré* equations

$$R^*_{f_{kk+1}} \ell'(f_{kk+1}) - L^*_{f_{k-1k}} \ell'(f_{k-1k}) = 0 \tag{2.3}$$

for $k = 1, \ldots, N-1$, where R^*_f and L^*_f for $f \in G$ are the right and left pull-backs by f, respectively, defined as follows: for $\alpha_g \in T^*_g G$, $R^*_f \alpha_g \in \mathfrak{g}^*$ is given by $\langle R^*_f \alpha_g, \xi \rangle = \langle \alpha_g, TR_f \xi \rangle$ for any $\xi \in \mathfrak{g}$, where TR_f is the tangent map of the right translation map $R_f : G \to G$; $h \to hf$, with a similar definition for L^*_f. Also, $\ell' : G \to T^*G$ is the differential of ℓ defined as follows. Let g^ϵ be a smooth curve in G such that $g^0 = g$ and $(\mathrm{d}/\mathrm{d}\epsilon)|_{\epsilon=0} g^\epsilon = v$. Then

$$\ell'(g)v = \left.\frac{\mathrm{d}}{\mathrm{d}\epsilon}\right|_{\epsilon=0} \ell(g^\epsilon).$$

For the other choice of the quotient in (2.2) given by $h_{k+1k} \equiv g_{k+1} g_k^{-1}$, the DEP equations are

$$L^*_{h_{k+1k}} \ell'(h_{k+1k}) - R^*_{h_{kk-1}} \ell'(h_{kk-1}) = 0. \tag{2.4}$$

Remark 2.2. *In the case that $\mathbb{L}$ is left invariant, the DEP equations take the form*

$$L^*_{f_{k+1k}} \ell'(f_{k+1k}) - R^*_{f_{kk-1}} \ell'(f_{kk-1}) = 0, \tag{2.5}$$

where $f_{k+1k} \equiv g_{k+1}^{-1} g_k$ is in the left quotient $(G \times G)/G$.

Notice that Eqs. (2.4) and (2.5) are formally the same.

We may associate to any C^1 function F defined on a neighborhood $\mathcal{V}$ of $\Delta \subset G \times G$ its Hamiltonian vector field X_F on $\mathcal{V} \supset \Delta$ satisfying $X_F \lrcorner\, \omega_{\mathbb{L}} = \mathrm{d}F$, where $\mathrm{d}F$, the differential of F, is a 1-form. The symplectic structure $\omega_{\mathbb{L}}$ naturally defines a Poisson structure on a neighborhood $\mathcal{V}$ of Δ (which we shall denote $\{\cdot,\cdot\}_{G\times G}$) by the usual relation

$$\{F, H\}_{G\times G} = \omega_{\mathbb{L}}(X_F, X_H).$$

Theorem 2.2 of [7] states that if the action of G on $G \times G$ is proper, the algorithm on G defined by the DEP equations (2.3) preserves the induced Poisson structure $\{\cdot,\cdot\}_G$ on $\mathcal{U} \subset G$ given by

$$\{f, h\}_G \circ \pi_{\mathrm{d}} = \{f \circ \pi_{\mathrm{d}}, h \circ \pi_{\mathrm{d}}\}_{G\times G} \tag{2.6}$$

for any C^1 functions f, h on $\mathcal{U}$, where $\mathcal{U} = \pi_{\mathrm{d}}(\mathcal{V})$.

Using the definition $f_{kk+1} = g_k g_{k+1}^{-1}$, the DEL algorithm can be reconstructed from the DEP algorithm by

$$(g_{k-1}, g_k) \mapsto (g_k, g_{k+1}) = (f_{k-1k}^{-1} g_{k-1}, f_{kk+1}^{-1} g_k), \tag{2.7}$$

where f_{kk+1} is the solution of (2.3). Indeed, $f_{kk+1}^{-1} g_k$ is precisely g_{k+1}. Similarly one shows that in the case of a left G action, the reconstruction of the DEP equation (2.5) is given by

$$(g_{k-1}, g_k) \mapsto (g_k, g_{k+1}) = (g_{k-1} f_{kk-1}^{-1}, g_k f_{k+1k}^{-1}).$$

2.2. The DLP algorithm

In addition to reconstructing the dynamics on $G \times G$, one may use the coadjoint action to form a *discrete Lie–Poisson* algorithm approximating the dynamics on $\mathfrak{g}^*$ [7]

$$\mu_{k+1} = \mathrm{Ad}^*_{f_{kk+1}} \mu_k, \tag{2.8}$$

where $\mu_k := \mathrm{Ad}^*_{g_k^{-1}} \mu_0$ is an element of the dual of the Lie algebra, μ_0 is the constant of motion (the momentum map value), and the sequence $\{f_{kk+1}\}$ is provided by the DEP algorithm on G.

The corresponding DLP equation for the left invariant system is given by

$$\Pi_{k+1} = \mathrm{Ad}^*_{f^{-1}_{k+1k}} \Pi_k, \tag{2.9}$$

where $\Pi_k := \mathrm{Ad}^*_{g_k} \pi_0$ and π_0 is the constant momentum map value. Henceforth, we shall use the notation $\mu \in \mathfrak{g}^*$ for the *right* invariant system and $\Pi \in \mathfrak{g}^*$ for the *left*.

3. Poisson structure and invariant manifolds on Lie groups

Discretization of an Euler–Poincaré system on TG results in a system on $G \times G$ defined by a Lagrangian $\mathbb{L}$. If it is regular, the Legendre transformation (in the sense of Veselov) $F\mathbb{L}$ define a symplectic form (and, hence, a Poisson structure) on $\mathcal{V} \subset G \times G$ via the pull-back of the canonical form from T^*G. Then, general Poisson reduction applied to these discrete settings defines a Poisson structure on the reduced space $\mathcal{U} = \pi_{\mathrm{d}}(\mathcal{V}) \subset G$. This approach was adopted in Theorem 2.2 of Ref. [7].

Alternatively, without appealing to the reduction procedure, a Poisson structure on a Lie group can be defined using ideas of Weinstein [12] on Lagrangian mechanics on groupoids and their algebroids. The key idea can be summarized in the following statements. A smooth function on a groupoid defines a natural (Legendre type) transformation between the groupoid and the dual of its algebroid. This transformation can be used to pull-back a canonical Poisson structure from the dual of the algebroid, provided the regularity conditions are satisfied.

The ideas outlined in this section can be easily expressed using the groupoid–algebroid formalism. Such a formalism is suited to the discrete gauge field theory generalization as well as to discrete semidirect product theory; nevertheless, the theory of groupoids and algebroids is not essential for the derivations, but rather contributes nicely to the elegance of the exposition.

3.1. Dynamics on groupoids and algebroids

In this section, we show that our discrete reduction methodology is consistent with Weinstein's groupoid–algebroid construction; the contents of this section are not essential for the remainder of the paper.

We briefly summarize results from Weinstein [12] and refer the reader to the original paper for details of proofs and definitions. Let Γ be a groupoid over a set M, with α, β :

$\Gamma \to M$ being its source and target maps, with a multiplication map $m : \Gamma_2 \to \Gamma$, where $\Gamma_2 \equiv \{(g, h) \in \Gamma \times \Gamma \,|\, \beta(g) = \alpha(h)\}$. Denote its corresponding algebroid by $\mathcal{A}$.

The Lie groupoids relevant to our exposition are the Cartesian product $G \times G$ of a Lie group G, with multiplication $(g, h)(h, k) = (g, k)$, and the group G itself. The corresponding algebroids are the tangent bundle TG and the Lie algebra $\mathfrak{g}$, respectively. The dual bundle to a Lie algebroid carries a natural Poisson structure. This is the Poisson bracket associated to the canonical symplectic form on T^*G and the Lie–Poisson structure on $\mathfrak{g}^*$, respectively.

Lagrangian mechanics on a groupoid Γ is defined as follows. Let $\mathcal{L}$ be a smooth, real-valued function on Γ, $\mathcal{L}_2$ the restriction to Γ_2 of the function $(g, h) \mapsto \mathcal{L}(g) + \mathcal{L}(h)$.

Definition 3.1. Let $\Sigma_{\mathcal{L}} \subset \Gamma_2$ be the set of critical points of $\mathcal{L}_2$ along the fibers of the multiplication map m; i.e. the points in $\Sigma_{\mathcal{L}}$ are stationary points of the function $\mathcal{L}(g)+\mathcal{L}(h)$ when g and h are restricted to admissible pairs with the constraint that the product gh is fixed [12].

A *solution of the Lagrange equations* for the Lagrangian function $\mathcal{L}$ is a sequence $\dots, g_{-2}, g_{-1}, g_0, g_1, g_2, \dots$ of elements of Γ, defined on some "interval" in $\mathbb{Z}$, such that $(g_j, g_{j+1}) \in \Sigma_{\mathcal{L}}$ for each j.

The Hamiltonian formalism for discrete Lagrangian systems is based on the fact that each Lagrangian submanifold of a symplectic groupoid determines a Poisson automorphism on the base Poisson manifold. Recall that the cotangent bundle $T^*\Gamma$ is, in addition to being a symplectic manifold, a groupoid itself, the base being $\mathcal{A}^*$; notice that both manifolds are naturally Poisson. The source and target mappings $\tilde{\alpha}, \tilde{\beta} : T^*\Gamma \to \mathcal{A}^*$ are induced by α and β.

Definition 3.2. Given any smooth function $\mathcal{L}$ on Γ, a Poisson map $\Lambda_{\mathcal{L}}$ from $\mathcal{A}^*$ to itself, which may be said to be generated by $\mathcal{L}$ is defined by the Lagrangian submanifold $\mathrm{d}\mathcal{L}(\Gamma)$ (under a suitable hypothesis of nondegeneracy) [12].

The appropriate "Legendre transformation" $F\mathcal{L}$ in the groupoid context is given by $\tilde{\alpha} \circ \mathrm{d}\mathcal{L} : \Gamma \to \mathcal{A}^*$ or $\tilde{\beta} \circ \mathrm{d}\mathcal{L} : \Gamma \to \mathcal{A}^*$, depending on whether we consider right or left invariance (through the definition of maps $\tilde{\alpha}$ and $\tilde{\beta}$). The transformation $F\mathcal{L}$ relates the mapping on Γ defined by $\Sigma_{\mathcal{L}}$ with the mapping $\Lambda_{\mathcal{L}}$ on $\mathcal{A}^*$. $F\mathcal{L}$ also pulls back the Poisson structure from $\mathcal{A}^*$ to Γ, which, in general, is defined only *locally* on some neighborhood $\mathcal{U} \subset \Gamma$. In the context of a Lie group, this means that any regular function $\ell : G \to \mathbb{R}$ defines a Poisson structure on $\mathcal{U}$. We shall address this issue in Sections 3.2 and 3.3. The reader is referred to [12] for an application of the above ideas to the groupoid $M \times M$ when the manifold M does not necessarily have group structure.

3.2. DEP equations as generators of Lie–Poisson Hamilton–Jacobi equations

A Lie group G is the simplest example of a groupoid with the base being just a point. Its algebroid is the corresponding Lie algebra $\mathfrak{g}$, with the dual being $\mathfrak{g}^*$. Consider left invariance and let a general function $\mathcal{L}$ on the group be specified by the discrete reduced Lagrangian

$\ell : G \to \mathbb{R}$. Then, the Legendre transform defined above is given by

$$F\ell = L_g^* \circ \mathrm{d}\ell : G \to \mathfrak{g}^*,$$

where $\mathrm{d}\ell : G \to T^*G$. Using these transformations we define

$$\Pi_{k-1} \equiv F\ell(f_{kk-1}) = L_{f_{kk-1}}^* \circ \mathrm{d}\ell(f_{kk-1}).$$

Recall the DEP equation (2.5) for left invariant systems

$$L_{f_{k+1k}}^* \mathrm{d}\ell(f_{k+1k}) - R_{f_{kk-1}}^* \mathrm{d}\ell(f_{kk-1}) = 0,$$

where we have identified the notations ℓ' and $\mathrm{d}\ell$. The latter equation can be rewritten as a system

$$\begin{aligned} \Pi_k &= L_f^* \circ \mathrm{d}\ell(f), \\ \Pi_{k+1} &= R_f^* \circ \mathrm{d}\ell(f), \end{aligned} \tag{3.1}$$

where the first equation is to be solved for f (which stands for f_{k+1k}) which then is substituted into the second equation to compute Π_{k+1}.

This system is precisely the Lie–Poisson Hamilton–Jacobi system described in Ref. [1] with the reduced discrete Lagrangian ℓ playing the role of the generating function. This means that there is no need to find an approximate solution of the reduced Hamilton–Jacobi equation [1]. Notice also that the DLP equation (2.9) is a direct consequence of the system (3.1)

$$\Pi_{k+1} = \mathrm{Ad}^*_{f_{k+1k}^{-1}} \Pi_k.$$

The following diagrams relate the dynamics on G and on $\mathfrak{g}^*$:

$$\begin{array}{ccccccc} G & \stackrel{\Sigma_\ell}{\to} & G & \qquad & f_{kk-1} & \stackrel{\Sigma_\ell}{\to} & f_{k+1k} \\ \downarrow F\ell & & \downarrow F\ell\,, & & \downarrow F\ell & & \downarrow F\ell\,, \\ \mathfrak{g}^* & \stackrel{\Lambda_\ell}{\to} & \mathfrak{g}^* & & \Pi_{k-1} & \stackrel{\Lambda_\ell}{\to} & \Pi_k \end{array} \tag{3.2}$$

where Σ_ℓ and Λ_ℓ are given in Definitions 3.1 and 3.2.

3.3. Some advantages of structure-preserving integrators

As we mentioned above, the "Legendre transform" $F\ell$ allows us to put a Poisson structure on the Lie group G, which, of course, depends on the discrete Lagrangian $\mathbb{L}$ on $G \times G$, and hence on the original Lagrangian L on TG (if we consider this from the discrete reduction point of view). It follows that the reduction of the discrete Euler–Lagrange dynamics on $G \times G$ is necessarily restricted to the symplectic leaves of this Poisson structure, so that these leaves are invariant manifolds, and correspond (under $F\ell$) to the symplectic leaves (coadjoint orbits) of the continuous reduced system on $\mathfrak{g}^*$.

These ideas are the content of Theorems 3.1 and 3.2.

Theorem 3.1. *Let L be a right-invariant Lagrangian on TG and let* $\mathbb{L}$ *be the Lagrangian of the corresponding discrete system on* $\mathcal{V} \subset G \times G$. *Assume that* $\mathbb{L}$ *is regular, in the sense that the Legendre transformation* $F\mathbb{L} : \mathcal{V} \to F\mathbb{L}(\mathcal{V}) \subset T^*G$ *is a local diffeomorphism, and let the quotient maps be given by*

$$\pi_{\mathrm{d}} : G \times G \to \frac{G \times G}{G} \cong G, \qquad \pi : T^*G \to \frac{T^*G}{G} \cong \mathfrak{g}^*.$$

Let ℓ *be the reduced Lagrangian on G defined by*

$$\mathbb{L} = \ell \circ \pi_{\mathrm{d}},$$

and let

$$F\ell : \mathcal{U} \subset G \to \mathfrak{g}^*$$

be the corresponding Legendre transform. Then the following diagram commutes:

$$\begin{array}{ccc} \mathcal{V} \subset G \times G & \overset{F\mathbb{L}}{\to} & T^*G \\ \downarrow\pi_{\mathrm{d}} & & \downarrow\pi \\ \mathcal{U} \subset G & \overset{F\ell}{\to} & \mathfrak{g}^* \end{array} . \tag{3.3}$$

Proof. First, we choose coordinate systems on each space. Let $(g_k, g_{k+1}) \in G \times G$ and $(g, p) \in T^*G$, so that the discrete quotient map (2.2) is given by $\pi_{\mathrm{d}} : (g_k, g_{k+1}) \mapsto f_{kk+1} = g_k g_{k+1}^{-1}$, and the continuous quotient map by $\pi : (g, p) \mapsto \mu = R_g^* p$. Recall that the fiber derivative $F\mathbb{L}$ in these coordinates has the following form (see, e.g., [13])

$$F\mathbb{L} : G \times G \to T^*G, \qquad (g_k, g_{k+1}) \mapsto (g_k, \mathrm{D}_1\mathbb{L}(g_k, g_{k+1})).$$

Then the above diagram (3.3) is given by

$$\begin{array}{ccc} (g_k, g_{k+1}) & \overset{F\mathbb{L}}{\to} & \left(g_k, p_k = \dfrac{\partial \mathbb{L}}{\partial g_k}\right) \\ \downarrow\pi_{\mathrm{d}} & & \downarrow\pi \\ f = R_{g_{k+1}^{-1}} g_k & & \mu = R_{g_k}^* p_k \end{array} , \tag{3.4}$$

where f stands for $f_{kk+1} = g_k g_{k+1}^{-1}$. To close this diagram and to verify the arrow determined by $F\ell$ compute the derivative of $\mathbb{L}$ using the chain rule

$$\mu = R_{g_k}^* p_k = R_{g_k}^* \frac{\partial(\ell \circ \pi)}{\partial g_k} = R_{g_k}^* \left(R_{g_{k+1}^{-1}}^* \frac{\partial \ell}{\partial f} \right) = R_f^* \frac{\partial \ell}{\partial f} = R_f^* \circ \ell'(f), \tag{3.5}$$

where we have used that, according to the definition of f, the partial derivative $\partial f / \partial g_k$ is given by the linear operator $TR_{g_{k+1}^{-1}}$. Eq. (3.5) is precisely the Legendre transformation $F\ell$ for a right-invariant system (see Section 3.2). □

Corollary 3.1. *Reconstruction of the DLP dynamics on* $\mathfrak{g}^*$ *by* π^{-1} *corresponds to the image of the DEL dynamics on* $G \times G$ *under the Legendre transformations* $F\mathbb{L}$ *and results in an* *algorithm on T^*G approximating the continuous flow of the corresponding Hamiltonian system*.

Proof. The proof follows from the results of Section 3.2, in particular, diagram (3.2) relates the DLP dynamics on $\mathfrak{g}^*$ with the DEP dynamics on $\mathcal{U} \subset G$ which, in turn, is related to the DEL dynamics on $\mathcal{V} \subset G \times G$ via the reconstruction (2.7). □

An important remark to Corollary 3.1 which follows from the results in Ref. [2] (see also [3]) is that, in general, to get a corresponding algorithm on the Hamiltonian side which is consistent with the corresponding continuous Hamiltonian system on T^*G, one must use the time step h-dependent Legendre transform given by the map

$$(g_k, g_{k+1}) \mapsto (g_k, -h\, \mathrm{D}_1\mathbb{L}(g_k, g_{k+1})).$$

The results of this paper are not affected, however, as we assume h to be constant and so we would simply add a constant multiplier to the corresponding symplectic and Poisson structures. For variable time-stepping algorithms, this remark is crucial and must be taken into account.

Theorem 3.2. *The Poisson structure on the Lie group G obtained by reduction of the Lagrange symplectic form $\omega_{\mathbb{L}}$ on $\mathcal{V} \subset G \times G$ via π_{d} coincides with the Poisson structure on $\mathcal{U} \subset G$ obtained by the pull-back of the Lie–Poisson structure ω_μ on $\mathfrak{g}^*$ by the Legendre transformation $F\ell$ (see diagram* (3.3)).

Proof. The proof is based on the commutativity of diagram (3.3) and the G invariance of the unreduced symplectic forms. Notice that G and $\mathfrak{g}^*$ in (3.3) are Poisson manifolds, each being foliated by symplectic leaves, which we denote Σ_f and $\mathcal{O}_\mu$ for $f \in G$ and $\mu \in \mathfrak{g}^*$, respectively. Denote by ω_f and ω_μ the corresponding symplectic forms on these leaves. We shall prove the compatibility of these structures under the diagram (3.3). Repeating this proof leaf-by-leaf establishes then the equivalence of the Poisson structures and proves the theorem.

Recall that the Lagrange 2-form $\omega_{\mathbb{L}}$ on $\mathcal{V} \subset G \times G$ derived from the variational principle coincides with the pull-back of the canonical 2-form ω_{can} on T^*G (see, e.g., [6,13]). Recall also that for a right-invariant system, reduction of T^*G to $\mathfrak{g}^*$ is given by right translation to the identity $e \in G$, i.e. any $p \in T_g^*G$ is mapped to $\mu = R_g^* p \in \mathfrak{g}^* \cong T_e^*G$. Thus, for any $g \in \pi^{-1}(\mu)$, where $\mu \in \mathfrak{g}^*$,

$$\pi^{-1}|_{T^*G} = R_{g^{-1}}^* : \mathfrak{g}^* \to T_g^*G,$$

so that $(\pi^{-1})^* = (R_{g^{-1}}^*)^*$ pulls back ω_{can} to ω_μ. Henceforth, π^{-1} shall denote the inverse map of π restricted to T_g^*G.

Let us write down using the above notations how the symplectic forms are being mapped under the transformations in diagram (3.3); we see that

$$\begin{array}{ccccccc} \mathcal{V} \subset G \times G & \stackrel{F\mathbb{L}}{\rightarrow} & T^*G & & \omega_{\mathbb{L}} & \stackrel{F\mathbb{L}^*}{\leftarrow} & \omega_{\text{can}} \\ \downarrow \pi_{\text{d}} & & \downarrow \pi, & & & & \uparrow (\pi^{-1})^* . \\ \mathcal{U} \subset G & \stackrel{F\ell}{\rightarrow} & \mathfrak{g}^* & & \omega_f & \stackrel{F\ell^*}{\leftarrow} & \omega_\mu \end{array} \tag{3.6}$$

Then, using the coordinate notations of diagram (3.4) for any $f \in G$ and $u, v \in T_f \Sigma_f$,

$$\omega_f(f)(u, v) \equiv \omega_\mu(\mu)(TF\ell(u), TF\ell(v)), \tag{3.7}$$

where $\mu = F\ell(f) \in \mathfrak{g}^*$. Continuing Eq. (3.7) using diagram (3.6), we have

$$\begin{aligned} \omega_f(f)(u, v) &= \omega_{\text{can}}((g_k, p_k))(T\pi^{-1} \circ TF\ell(u), T\pi^{-1} \circ TF\ell(v)) \\ &= \omega_{\mathbb{L}}((g_k, g_{k+1}))(TF\mathbb{L}^{-1} \circ T\pi^{-1} \circ TF\ell(u), TF\mathbb{L}^{-1} \circ \\ &\quad T\pi^{-1} \circ TF\ell(v)), \end{aligned} \tag{3.8}$$

where $(g_k, p_k) \in \pi^{-1}(\mu)$ and $T\pi^{-1}$ denotes $TR^*_{g^{-1}}$.

Using (3.3), it follows that

$$F\ell \circ \pi_{\text{d}} = \pi \circ F\mathbb{L}$$

and, hence, for the tangent maps, we have

$$TF\ell \circ T\pi_{\text{d}} = T\pi \circ TF\mathbb{L}.$$

So, if u, v in (3.7) are images of some G-invariant vector fields U, V on $\mathcal{V} \subset G \times G$, i.e. $u = T\pi_{\text{d}}(U)$, $v = T\pi_{\text{d}}(V)$, then from (3.8) it follows that

$$\omega_f(f)(u, v) = \omega_{\mathbb{L}}((g_k, g_{k+1}))(U, V),$$

where $(g_k, g_{k+1}) = \pi_{\text{d}}^{-1}(f)$ and $U, V \in T_{(g_k, g_{k+1})} G \times G$. The last equation precisely means that ω_f is the discretely reduced symplectic form, i.e. the image of $\omega_{\mathbb{L}}$ under the quotient map π_{d}. □

Analogous theorems hold for the case of left invariant systems.

More general configuration spaces. Similar ideas carry over to the integration of systems defined on a general configuration space M with some symmetry group G. In this case, the reduced discrete space $(M \times M)/G$ inherits a Poisson structure from the one defined on $M \times M$ (analogously to (2.6)). Its symplectic leaves again become dynamically invariant manifolds for structure-preserving integrators and can be viewed as images of the symplectic leaves of the reduced Poisson manifold T^*M/G under appropriately defined "Legendre transformations". This is a topic of ongoing research that builds on recent progress in Lagrangian reduction theory (see [8]).

3.4. Poisson structures of the rigid body

As an example of applications of the above ideas, we consider the dynamics of the rigid body and its associated reduction and discretization (see, e.g. [4,5,7,9] for more details).

The basic set up. The configuration space of the system is SO(3). The corresponding Lagrangian is determined by a symmetric positive definite operator $J : \mathfrak{so}(3) \to \mathfrak{so}(3)$, defined by $J(\xi) = \Lambda\xi + \xi\Lambda$, where $\xi \in \mathfrak{so}(3)$ and Λ is a diagonal matrix satisfying $\Lambda_i + \Lambda_j > 0$ for all $i \neq j$. The left invariant metric on SO(3) is obtained by left translating the bilinear form at $e \in \mathrm{SO}(3)$ which is given by

$$(\xi, \xi) = \tfrac{1}{4}\mathrm{Tr}(\xi^{\mathrm{T}} J(\xi)).$$

The operator J, viewed as a mapping $\mathcal{J} : \mathfrak{so}(3) \to \mathfrak{so}(3)^*$, has the usual interpretation of the inertia tensor, and the Λ_i correspond to the sums of certain principal moments of inertia.

The rigid body Lagrangian is the kinetic energy of the system

$$L(g, \dot{g}) = \tfrac{1}{4}\langle g^{-1}\dot{g}, \mathcal{J}(g^{-1}\dot{g})\rangle = \tfrac{1}{4}\langle \xi, \mathcal{J}\xi\rangle = l(\xi), \tag{3.9}$$

where $\xi = g^{-1}\dot{g} \in \mathfrak{so}(3)$ and $\langle\cdot,\cdot\rangle$ is the pairing between the Lie group and its dual.

Poisson structures and Casimir functions. The Lie algebra dual $\mathfrak{so}(3)^*$ has a well-known Lie–Poisson structure with a Casimir $C_{\mathfrak{so}(3)^*}(\mu) = \mathrm{Tr}(\mu^2)$, where $\mu \in \mathfrak{so}(3)^*$. Upon identification with $\mathbb{R}^3$, its generic symplectic leaves become concentric spheres with Kirillov–Kostant symplectic form being proportional to the area form. If y denotes coordinates on $\mathbb{R}^3 \cong \mathfrak{so}(3)^*$, then the Casimir function is given by $C_{\mathfrak{so}(3)^*}(y) = \|y\|^2$.

Following Section 5 of Ref. [7] on discrete Euler–Poincaré reduction, we obtain the reduced form of the Moser–Veselov Lagrangian on the group $\mathfrak{so}(3)$ given by

$$\ell(f) = \mathrm{Tr}(f\Lambda),$$

where $f \in \mathfrak{so}(3)$ and $\mathfrak{so}(3)$ is embedded into the linear space $\mathfrak{gl}(3)$. Then, the Legendre transform $F\ell$ takes the form

$$F\ell(f) = L_f^* \circ \mathrm{d}\ell(f) = \mathrm{skew}(f\Lambda) = f\Lambda - \Lambda f^{\mathrm{T}} : \mathfrak{so}(3) \to \mathfrak{so}(3)^*,$$

where the constraint that f be in $\mathfrak{so}(3)$ has been enforced. The pull-back of $C_{\mathfrak{so}(3)^*}$ under $F\ell^*$ defines a Casimir function *on the group*, which up to a constant term and a sign, is given by

$$C_{\mathfrak{so}(3)}(f) = \mathrm{Tr}(f\Lambda f\Lambda), \quad f \in \mathfrak{so}(3). \tag{3.10}$$

Its symplectic leaves constitute the invariant manifolds of the reduced discrete dynamics corresponding to the Lagrangian (3.9).

Analogously, one can define a Poisson structure on the Lie algebra $\mathfrak{so}(3)$ using the duality between Lie–Poisson and Euler–Poincaré reduced systems on $\mathfrak{so}(3)^*$ and $\mathfrak{so}(3)$, respectively. The Lagrangian (3.9) defines the Legendre transformations *Fl* from $\mathfrak{so}(3)$ to $\mathfrak{so}(3)^*$ given by $\mu = \partial l/\partial\xi = \mathcal{J}(\xi)$. Then, the pull-back by Fl^* defines a Casimir function on $\mathfrak{so}(3)$:

$$C_{\mathfrak{so}(3)}(\xi) = Fl^* C_{\mathfrak{so}(3)^*}(\xi) = \langle\langle \mathcal{J}(\xi), \mathcal{J}(\xi)\rangle\rangle,$$

where the metric on the dual is induced by the one on the algebra, i.e. by the symmetric positive definite operator J. If x denotes coordinates on $\mathbb{R}^3 \cong \mathfrak{so}(3)$, then the Casimir function is given by $C_{\mathfrak{so}(3)}(x) = \|\mathcal{J}(x)\|^2$. Thus, the corresponding symplectic leaves are ellipsoids of $\mathcal{J}^2$. They *do not* coincide with adjoint orbits, which are spheres in $\mathbb{R}^3$. The dynamic orbits are obtained by intersecting these ellipsoids, determined by $\mathcal{J}^2$, with the energy ellipsoids, determined by $\mathcal{J}$.

Acknowledgements

The authors would like to thank Alan Weinstein for pointing out the connections with the general theory of dynamics on groupoids and algebroids and Sameer Jalnapurkar for comments and discussions. SP and SS would also like to thank the Center for Nonlinear Science in Los Alamos for providing a valuable setting for part of this work. SS was partially supported by NSF-KDI grant ATM-98-73133, and JEM and SP were partially supported by NSF-KDI grant ATM-98-73133 and the Air Force Office of Scientific Research.

References

[1] Z. Ge, J.E. Marsden, Lie–Poisson Hamilton–Jacobi theory and Lie–Poisson integrators, Phys. Lett. A 133 (1988) 134–139.
[2] C. Kane, J.E. Marsden, M. Ortiz, Symplectic-energy–momentum preserving variational integrators, J. Math. Phys. 40 (1999) 3353–3371.
[3] C. Kane, J.E. Marsden, M. Ortiz, M. West, Variational Integrators and the Newark Algorithm for Conservative and Dissipate Mechanical Systems, Int. J. Num. Meth. Eng., to appear.
[4] D. Lewis, J.C. Simo, Conserving algorithms for the *N*-dimensional rigid body, Fields Inst. Commun. 10 (1996) 121–139.
[5] J.E. Marsden, T.S. Ratiu, Introduction to Mechanics and Symmetry, 2nd Edition, Springer, Berlin, 1999, p. 1994.
[6] J.E. Marsden, G. Patrick, S. Shkoller, Multisymplectic geometry, variational integrators, and nonlinear PDEs, Commun. Math. Phys. 199 (1998) 351–395.
[7] J.E. Marsden, S. Pekarsky, S. Shkoller, Discrete Euler–Poincaré and Lie–Poisson equations, Nonlinearity 12 (1999) 1647–1662.
[8] J.E. Marsden, T.S. Ratiu, J. Scheurle, Reduction theory and the Lagrange–Routh equations, J. Math. Phys., in press.
[9] J. Moser, A.P. Veselov, Discrete versions of some classical integrable systems and factorization of matrix polynomials, Commun. Math. Phys. 139 (1991) 217–243.
[10] A.P. Veselov, Integrable discrete-time systems and difference operators, Funk. Anal. Prilozhen. 22 (1988) 1–13.
[11] A.P. Veselov, Integrable Lagrangian correspondences and the factorization of matrix polynomials, Funk. Anal. Prilozhen. 25 (1991) 38–49.
[12] A. Weinstein, Lagrangian mechanics and groupoids, Fields Inst. Commun. 7 (1996) 207–231.
[13] J.M. Wendlandt, J.E. Marsden, Mechanical integrators derived from a discrete variational principle, Physica D 106 (1997) 223–246.